2015

(October)

MATHEMATICS

(Elective/Honours)

FIRST PAPER

(Algebra—I and Calculus—I)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** question, taking **one** from each Unit

UNIT—I

1. *(a)* If a finite set S has n elements, then prove that the power set of S has 2^n elements. 4

(b) Show that the domain of definition of the function $f(x) = \log \frac{1-x}{1+x}$ is the interval $(-1, 1)$. Also, show that for $x_1, x_2 \in (-1, 1)$

$$f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$$ 2+3

(c) Draw the graph of the function

$$f(x)=\begin{cases}0\ , & x=0\\ 1-x, & 0<x<1\\ 1\ , & x=1\end{cases}$$

The graph suggests that the function $f(x)$ assumes at least once every value between $f(0)$ and $f(1)$ and yet it has a discontinuity in [0, 1]. Justify this. 3+3

2. (a) Let $S=\{-2, -1, 0, 1, 2\}$. Give examples of the following : 2+2+2

(i) A relation on S that is reflexive but not transitive

(ii) A relation on S that is reflexive but not symmetric

(iii) A relation on S that is symmetric but not reflexive

(b) Using definition of limit, prove that

$$\lim_{x\to 2} 5x=10$$ 3

(c) Discuss the continuity of the function $f(x)=\dfrac{|x|}{x}$ at $x=0$ and $x=1$. 3+3

UNIT—II

3. (a) Let $f: Q\to Q$ be defined by $f(x)=2x+3$, where $Q=$ set of rational numbers. Show that f is one-to-one and onto. Also, find a formula that defines the inverse function f^{-1}. 3+2

(b) Give examples of—

(i) matrices A, B such that $AB\neq BA$;

(ii) matrices A, B such that $AB=0$ but $A\neq 0$, $B\neq 0$. 2+2

(c) Determine if the following system of equations is consistent and if so, find the solution : 2+4

$$x-y+2z=4$$
$$3x+y+4z=6$$
$$x+y+z=1$$

4. (a) If B is an idempotent matrix, show that $A=I-B$ is also idempotent and $AB=BA=0$. 3+2

(b) Consider the equation

$$f(x)\equiv x^3-3x^2+4x-3=0$$

Find $f(A)$ and A^{-1} if

$$A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$$ 4

(c) Applying elementary row operation, find the rank of the matrix

$$\begin{pmatrix} -4 & 1 & -1 & 2 \\ 1 & 0 & -1 & 0 \\ -5 & 2 & -5 & 4 \end{pmatrix}$$ 6

UNIT—III

5. (a) Show that $f(x) = x^2$ for $-2 \le x \le 2$ is uniformly continuous. 4

(b) Find the slopes of the parabola $y = x^2$ at the vertex and at the point $\left(\frac{1}{2}, \frac{1}{4}\right)$. Determine whether the tangent line to the parabola at the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ makes an angle $45°$ with x-axis with justification. 3+2

(c) If $y = \cos(m \sin^{-1} x)$, show that—

(i) $(1-x^2)y_2 - xy_1 + m^2 y = 0$;

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$. 2+4

6. (a) If the rate of change of y with respect to x is 5 and x is changing at 3 units per second, how fast is y changing? 3

(b) Find $\frac{d^2y}{dx^2}$, where $y = x\sqrt{x^2+9}$ at $x = 4$. 3

(c) Find the nth derivative of $\sqrt{x}$. 4

(d) Evaluate any *two* of the following : 2½×2=5

(i) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(ii) $\lim_{x \to 0} x \log x$

(iii) $\lim_{x \to 0} x^{2 \sin x}$

UNIT—IV

7. (a) Integrate any *two* of the following : 4×2=8

(i) $\int \frac{x^2}{1-x^4} dx$

(ii) $\int \log(1+x)^{(1+x)} dx$

(iii) $\int \frac{dx}{(1-x)\sqrt{1+x}}$

(b) Express

$$\int_a^b x^2 \, dx$$

as the limit of a sum and evaluate it. 4

(c) Use the properties of definite integral to show that

$$\int_0^{\pi/2} \log \tan x \, dx = 0$$ 3

8. *(a)* Prove that

$$\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(x)\,dx$$ 2

(b) Evaluate :

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$$ 4

(c) Obtain reduction formula for

$$\int_0^{\pi/2} \cos^n x\,dx$$

n being a positive integer greater than 1.
Hence, evaluate

$$\int_0^1 x^2\sqrt{1-x^2}\,dx$$ 3+3

(d) Show that

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2}$$ 3

UNIT—V

9. *(a)* State the degree and order of the following differential equations : 1+1

(i) $\dfrac{d^2y}{dx^2} = x^3\left(\dfrac{dy}{dx}\right)^2$

(ii) $y\left(\dfrac{dy}{dx}\right)^3 + 2x\dfrac{dy}{dx} - y = 0$

(b) Obtain the differential equation of the system of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

(λ is a variable parameter)

and hence, show that the system is self-orthogonal. 3+3

(c) Solve any *two* of the following : 3½×2=7

(i) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

(ii) $x\,dy - y\,dx + a(x^2+y^2)\,dx = 0$

(iii) $\sec^2 y\dfrac{dy}{dx} + 2x\tan y = x^3$

10. *(a)* Find the general and singular solutions of $y = px + \sqrt{4p^2+1}$. 4

(b) Find $f(x)$, if $f'(x) = x\,f(x)$ and $f(0) = 1$. 3

(c) Find the equation of the curve whose slope at any point (x, y) on it is xy and which passes through the point (0, 1). 3

(d) Solve any *one* of the following : 5

(i) $(D^2 - 9D + 20)y = x^2 e^{3x}$

(ii) $(D^2 - D - 2)y = \sin 2x$

(iii) $(D^2 - 2D + 1)y = 0;\ \ y = 0,\ \ Dy = 1$ when $x = 0$

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