5/H-29 (v) (Syllabus-2015)

2019

(October)

MATHEMATICS

(Honours)

(GHS-51)

(Elementary Number Theory and Advanced Algebra)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

Answer Elementary Number Theory and Advanced Algebra in two separate books.

(ELEMENTARY NUMBER THEORY)

UNIT—I

1. (a) State whether the following statements are True or False with brief justification (any *five*) : (a, b, c, n denote integers) 2×5=10

   (i) If $a \mid bc$ and $\gcd(a, b) \neq 1$, then $a \mid c$.

   (ii) If n is an odd integer, then $\gcd(n, n+2) = 1$.

   (iii) If $a \mid bc$, then $a \mid b$ or $a \mid c$.

(Turn Over)

(iv) If a and b are odd integers, then $a^2 - b^2$ is divisible by 8.

(v) If

$$\gcd(a, p^2) = p \text{ and } \gcd(b, p^3) = p^2$$

where p is a prime, then

$$\gcd(ab, p^4) = p^2$$

(vi) If p is a prime and

$$p \mid (a^2 + b^2) \text{ and } p \mid (b^2 + c^2)$$

then $p \mid (a^2 - c^2)$.

(vii) If $a^3 \mid c^3$, then $a \mid c$.

(b) Find integers x and y which satisfy

$$\gcd(28, 72) = 28x + 72y$$ 3

(c) Exhibit ten consecutive integers such that all are composite integers. 2

2. (a) State and prove Euler's theorem. 1+5=6

(b) If p is an odd prime, then prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod p$$ 4

(c) Show that the set

$$\{0, -7, 10, -5, 12, -3, 14, -1\}$$

is a complete residue system modulo 8. 3

(d) Prove that $n^2 \equiv 1 \pmod 8$ for any odd integer n. 2

UNIT—II

3. (a) State and prove Chinese remainder theorem. 1+5=6

(b) If m and n are relatively prime positive integers, then prove that

$$\phi(m \cdot n) = \phi(m) \cdot \phi(n)$$ 5

(c) Solve the linear congruence

$$6x \equiv 15 \pmod{21}$$ 4

4. (a) Let x and y be real numbers. Then prove that

$$[x] + [y] \le [x + y] \le [x] + [y] + 1$$ 4

(b) State and prove Mobius inversion formula. 1+4=5

(c) Compute $\sigma(n)$ and $\tau(n)$ for $n = 3600$. 2+2=4

(d) Prove that if $n = 2^k$ for some integer $k \ge 1$, then prove that $\phi(n) = \frac{n}{2}$. 2

(ADVANCED ALGEBRA)

UNIT—III

5. (a) Define normal subgroup. Give an example. 1+1=2

(b) Prove that a subgroup N of a group G is normal if

$$(xN)(yN) = (xy)N, \quad \forall x, y \in G$$ 3

(Turn Over)

(c) Prove that if K is a subgroup of a group G such that $g^2 \in K, \ \forall g \in G$, then K is normal in G. 3

(d) Prove that S_4 does not have a normal subgroup of order 3. 3

(e) Let G be a group, for $g \in G$, define $T_g : G \to G$ by $T_g(x) = gxg^{-1}, \ \forall x \in G$. Prove that T_g is an automorphism of G. 4

6. *(a)* Define integral domain. Prove that

$$\mathbb{Z}_n = \{0, 1, 2, \cdots, (n-1)\}$$

w.r.t. addition and multiplication modulo n is not an integral domain if n is not a prime. 1+3=4

(b) Define an ideal of a ring. Prove that

$$\mathbb{Z} = 98\mathbb{Z} + 99\mathbb{Z}$$

1+2=3

(c) Let R be a ring with a unit element such that $a^2 = a, \ \forall a \in R$. Prove that—

(i) R is commutative;

(ii) every prime ideal of R is maximal. 4+4=8

UNIT—IV

7. *(a)* Prove that a finite integral domain is a field. 6

(b) Prove that any non-zero ring homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is identity. 4

(c) What is a principal ideal domain (**PID**)? Is $\mathbb{Z}[x]$ a **PID**? Justify. 1+4=5

8. *(a)* Prove that if F is a field, then $F[x]$ is a Euclidean ring. 7

(b) Prove that

$$x^3 + x^2 + 1$$

is irreducible in $\mathbb{Z}_5[x]$. 2

(c) Prove that

$$\frac{\mathbb{Z}_5[x]}{\langle x^3 + x^2 + 1 \rangle}$$

is a field. 3

(d) How many elements are there in

$$\frac{\mathbb{Z}_5[x]}{\langle x^3 + x^2 + 1 \rangle}?$$

Justify. 3

UNIT—V

9. *(a)* Prove that a subset W of a vector space V over a field F is a subspace of V if and only if $\forall a, b \in F$ and $u, v \in W$, $au + bv \in W$. Hence show that the set

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y + z = 0\}$$

is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$. 5+3=8

(b) If

$$U = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}$$
$$\text{and } W = \{(x, y) \in \mathbb{R}^2 \mid y = 3x\}$$

then show that $U \oplus W = V$ where $V = \mathbb{R}^2$, a vector space over the field $\mathbb{R}$. 4

(c) Let $B = \{v_1, v_2, \cdots, v_n\}$ be a basis of a finite dimensional vector space V over a field F. Show that every vector $v \in V$ can be expressed uniquely as a linear combination of vectors in B. 3

10. *(a)* Show that the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + y, y + z, z + x)$$

is a linear transformation, where $\mathbb{R}^3$ is a vector space over $\mathbb{R}$. 3

(b) Let U and V be vector spaces over a field F. If $T : U \to V$ is a linear transformation, then show that—

(i) $\ker(T)$ is a subspace of U;

(ii) T is one-one if and only if $\ker(T) = \{0\}$, where $\ker(T)$ is the kernel of T. 2+4=6

(c) Consider the linear operator

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

defined by

$$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$$

Show that T is invertible and find T^{-1}. 3+3=6

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