2019

(October)

MATHEMATICS

(Elective/Honours)

(GHS–31)

(Algebra—II & Calculus—II)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Show that the set of all positive rational numbers forms an Abelian group under the composition defined by

$$a * b = \frac{ab}{2}$$ 5

(b) If the inverse of a is a^{-1}, then show that the inverse of $a^{-1} = a$. 2

(c) Show that the set consisting of the fourth roots of unity namely $1, -1, i, -i$ form a group with respect to multiplication. 5

(d) Show that if every element of a group G is its own inverse, then G is Abelian. 3

2. (a) Show that if a, b are any two elements of a group G, then $(ab)^2 = a^2b^2$ if and only if G is Abelian. 3

(b) Give an example to show that the union of two subgroups is not necessarily a subgroup. 2

(c) Prove that any two right (left) cosets of a subgroup are either disjoint or identical. 5

(d) Prove that every group of prime order is cyclic. 5

UNIT—II

3. (a) Prove that

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

n being any positive integer. 5

(b) Solve the following equation by Cardan's method : 5

$$x^3 + 6x + 7 = 0$$

(Continued)

(c) Solve the equation

$$x^4 - 3x^3 - 5x^2 + 9x - 2 = 0$$

if one of the roots of the given equation is $2+\sqrt{3}$. 5

4. (a) If the sum of the two roots of

$$x^4 + px^3 + qx^2 + rx + s = 0$$

is equal to the sum of its other two roots, then prove that $p^3 + 8r = 4pq$. 5

(b) If

$$x^4 - 14x^2 + 24x - k = 0$$

has four real, unequal roots, prove that k must lie between 8 and 11. 5

(c) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are

$$\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}, \frac{\alpha+\beta}{\gamma^2}$$ 5

UNIT—III

5. (a) Prove that a sequence which is monotonic increasing and bounded above converges to its exact upper bound. 5

(Turn Over)

(b) Show that the sequence

$$\left\{1-\frac{1}{n}\right\}$$

is bounded above. Is it monotonic? Find its limit, if the limit exists. 5

(c) Prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence. 5

6. (a) Test the convergence of the following series (any *two*) : 3×2=6

(i) $\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\ldots$

(ii) $\frac{1}{1\cdot 2^2}+\frac{1}{2\cdot 3^2}+\ldots$

(iii) Σu_n, where $u_n=\frac{\sqrt{n}}{n^2-1}$

(b) By using Leibnitz's test, prove that the series

$$1-\frac{1}{2}+\frac{1}{3}-\ldots$$

is convergent. 3

(Continued)

(c) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{n^2}{\sqrt{n}}$$ 6

UNIT—IV

7. (a) State and prove Cauchy's mean value theorem. 1+4=5

(b) Verify Rolle's theorem for the function $f(x)=x^2-5x+6$ in $1\le x\le 4$. 3

(c) Find the maximum and minimum value of

$$1+2\sin x+3\cos^2 x,\quad 0\le x\le\frac{1}{2}\pi$$ 3

(d) Show that the rectangle inscribed in a circle has maximum area, when it is a square. 4

8. (a) If $u=\log r$, $r^2=x^2+y^2+z^2$, prove that

$$r^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$$ 6

(Turn Over)

(b) If $v = f(u)$, u being a homogeneous function of degree n in x and y, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = nu\frac{\partial v}{\partial u}$$ 4

(c) Show that

$$x^3 - 6x^2 + 12x - 3$$

is neither a maximum nor a minimum, when $x = 2$. 5

UNIT—V

9. (a) Expand $\cos x$ in powers of x with Maclaurin's form of remainder. 4

(b) State and prove the fundamental theorem of integral calculus. 6

(c) Find the length of the arc of the parabola $y^2 = 4ax$ measured from the vertex to one extremity of the latus rectum. 5

10. (a) Show that the volume of the solid formed by the rotation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about major axis is $\frac{4}{3}\pi ab^2$ and about minor axis is $\frac{4}{3}\pi a^2 b$. 4

(Continued)

(b) Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point $x = 2$. 5

(c) Evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} x\,dx\,dy$$ 6

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